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The anti-self-dual Yang–Mills equation and classical transcendental solutions to the Painlevé II and IV equations

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Abstract
We present a determinant expression for a family of solutions to the $SL(2, \mathbb{C})$ anti-self-dual Yang–Mills equation that corresponds to the classical transcendental solutions of the Painlevé II and Painlevé IV equations.

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1. Introduction

Both the anti-self-dual Yang–Mills (ASDYM) equation and the six Painlevé equations play a key role in the theory of integrable systems. Mason and Woodhouse have shown that the $SL(2, \mathbb{C})$ ASDYM equation defined on $\mathbb{C}^4$ can be reduced to the Painlevé equations under certain three-dimensional Abelian groups of conformal symmetries [1, 2]. Murata has reconstructed this reduction process by using the notions in the theory of generalized confluent hypergeometric functions in [3].

Corrigan *et al* have constructed a family of solutions to Yang's equation, which is equivalent to the ASDYM equation, in the case of $SL(2, \mathbb{C})$ [4, 5]. These solutions can be expressed in terms of Hankel determinants whose entries satisfy the Laplace equation. An approach in terms of τ-functions to Yang's equation and to the family of solutions has been proposed in [6].

On the other hand, it is known that the classical solutions to the Painlevé equations admit determinant expressions. In particular, the classical transcendental solutions can be expressed in terms of two-directional Wronskians whose entries satisfy (confluent) hypergeometric differential equations [7].

It is meaningful to investigate the reduction process from the ASDYM equation to the Painlevé equations with respect to special solutions and their τ-functions. The aim of this paper is to construct a family of solutions to the ASDYM equation and Yang's equation that corresponds to the classical transcendental solutions of the Painlevé II and IV equations.

In section 2, we give a brief review of Yang's equation and its special solutions. According to [3, 8, 9], we summarize the derivation of the Painlevé II equation from the ASDYM equation in section 3. In section 4, we discuss on the Riccati solution to P_{II}. We find that the Laplace equation, which characterizes a particular solution to Yang's equation, is reduced to Airy's differential equation. In section 5, we construct a family of solutions to Yang's equation that corresponds to the classical transcendental solutions of P_{II}. We give a similar discussion on the classical transcendental solutions to the Painlevé IV equation in section 6. Section 7 is devoted to a remark on the Bäcklund transformations.

2. Yang's equation and its determinant solutions

In this section, we give a brief review of Yang's equation [10], its symmetries and a family of special solutions expressed in terms of Hankel determinants [4, 5].

The $SL(2,\mathbb{C})$ ASDYM equation is given by

$$\begin{aligned}
&\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0,\\
&\partial_{\tilde z} A_{\tilde w} - \partial_{\tilde w} A_{\tilde z} + [A_{\tilde z}, A_{\tilde w}] = 0,\\
&\partial_z A_{\tilde z} - \partial_{\tilde z} A_z - \partial_w A_{\tilde w} + \partial_{\tilde w} A_w + [A_z, A_{\tilde z}] - [A_w, A_{\tilde w}] = 0,
\end{aligned} \tag{2.1}$$

where $A_{\tilde z}$, $A_{\tilde w}$, A_z and A_w are the components of the gauge potential $A = A_{\tilde z}\mathrm{d}\tilde z + A_{\tilde w}\mathrm{d}\tilde w + A_z\mathrm{d}z + A_w\mathrm{d}w$ and are $\mathfrak{sl}(2,\mathbb{C})$-valued functions. The first two equations of (2.1) are the local integrability conditions for the existence of two matrix-valued functions H and $\widetilde H$ such that

$$\partial_{\tilde z}\widetilde H = -A_{\tilde z}\widetilde H, \qquad \partial_{\tilde w}\widetilde H = -A_{\tilde w}\widetilde H, \qquad \partial_z H = -A_z H, \qquad \partial_w H = -A_w H. \tag{2.2}$$

They are determined uniquely by A up to $H \mapsto H\widetilde M$, $\widetilde H \mapsto \widetilde H M$, where M depends only on z and w, and $\widetilde M$ depends only on $\tilde z$ and $\tilde w$. The third equation of (2.1) holds if and only if the J*-matrix* defined by $J = \widetilde H^{-1}H$ satisfies Yang's equation

$$\partial_w(J^{-1}\partial_{\tilde w}J) - \partial_z(J^{-1}\partial_{\tilde z}J) = 0. \tag{2.3}$$

It is obvious that Yang's equation is invariant under the transformation

$$J \mapsto M^{-1}J\widetilde M, \tag{2.4}$$

which means that one can regard the transformation (2.4) as the Bäcklund transformation of Yang's equation (2.3). It is known that Yang's equation (2.3) also admits another Bäcklund transformation. We set

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2 + eg \end{pmatrix}, \tag{2.5}$$

to express Yang's equation (2.3) as the coupled nonlinear equations

$$\begin{aligned}
&\partial_z\partial_{\tilde z}(\log f) + \frac{(\partial_{\tilde z}e)(\partial_z g)}{f^2} = \partial_w\partial_{\tilde w}(\log f) + \frac{(\partial_{\tilde w}e)(\partial_w g)}{f^2},\\
&\partial_{\tilde z}\left(\frac{\partial_z g}{f^2}\right) = \partial_{\tilde w}\left(\frac{\partial_w g}{f^2}\right),\\
&\partial_z\left(\frac{\partial_{\tilde z} e}{f^2}\right) = \partial_w\left(\frac{\partial_{\tilde w} e}{f^2}\right).
\end{aligned} \tag{2.6}$$

Lemma 2.1. *Let (e, f, g) be a solution to (2.6). Then $(\hat{e}, \hat{f}, \hat{g})$ defined by*

$$\hat{f} = \frac{1}{f}, \qquad \partial_z \hat{g} = \frac{\partial_{\tilde{w}} e}{f^2}, \qquad \partial_w \hat{g} = \frac{\partial_{\tilde{z}} e}{f^2}, \qquad \partial_{\tilde{z}} \hat{e} = \frac{\partial_w g}{f^2}, \qquad \partial_{\tilde{w}} \hat{e} = \frac{\partial_z g}{f^2}, \tag{2.7}$$

is also a solution.

We call (2.7) the transformation β. Obviously, we have $\beta^2 = 1$.

As a particular example of the Bäcklund transformation (2.4), we consider the transformation γ defined by

$$\gamma: \qquad J \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} J \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \tag{2.8}$$

which acts on a solution to (2.6) according to

$$\gamma: \qquad f \mapsto \frac{f}{f^2 + eg}, \qquad g \mapsto \frac{e}{f^2 + eg}, \qquad e \mapsto \frac{g}{f^2 + eg}. \tag{2.9}$$

We also have $\gamma^2 = 1$. Since $\beta\gamma \neq \gamma\beta$, it is possible to generate solutions to Yang's equation by operating one after the other. In fact, Corrigan *et al* [4, 5] have constructed a family of solutions by applying these Bäcklund transformations to a particular solution characterized by the Laplace equation,

$$J = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix}, \qquad (\partial_w \partial_{\tilde{w}} - \partial_z \partial_{\tilde{z}})\varphi = 0. \tag{2.10}$$

Proposition 2.2. *[4, 5] Define the functions $\tau_n^m (m \in \mathbb{Z}, n \in \mathbb{Z}_{\geqslant 0})$ by*

$$\tau_n^m = \begin{vmatrix} \varphi_{m-n+1} & \varphi_{m-n+2} & \cdots & \varphi_m \\ \varphi_{m-n+2} & \varphi_{m-n+3} & \cdots & \varphi_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m+1} & \cdots & \varphi_{m+n-1} \end{vmatrix}, \tag{2.11}$$

where the entries φ_j satisfy

$$\partial_{\tilde{w}} \varphi_j = \partial_z \varphi_{j+1}, \qquad \partial_{\tilde{z}} \varphi_j = \partial_w \varphi_{j+1} \tag{2.12}$$

and the Laplace equation

$$(\partial_w \partial_{\tilde{w}} - \partial_z \partial_{\tilde{z}})\varphi_j = 0. \tag{2.13}$$

Then

$$J = J_{m,n} = \frac{1}{\tau_n^m} \begin{pmatrix} \tau_n^{m-1} & \tau_{n+1}^m \\ \tau_{n-1}^m & \tau_n^{m+1} \end{pmatrix} \tag{2.14}$$

gives rise to a family of solutions to Yang's equation (2.3).

The functions τ_n^m defined by (2.11) satisfy the bilinear relations [6]

$$\begin{aligned} &D_{\tilde{w}} \tau_n^m \cdot \tau_{n-1}^{m+1} = D_z \tau_n^{m+1} \cdot \tau_{n-1}^m, \\ &D_{\tilde{z}} \tau_n^m \cdot \tau_{n-1}^{m+1} = D_w \tau_n^{m+1} \cdot \tau_{n-1}^m, \\ &\tau_{n+1}^m \tau_{n-1}^m = \tau_n^{m+1} \tau_n^{m-1} - \tau_n^m \tau_n^m, \end{aligned} \tag{2.15}$$

where D is Hirota's bilinear operator. The first two relations of (2.15) are reduced to the sum of Plücker relations.

The above family of solutions are specified by two discrete parameters m and n which denote the label of the functions φ_j and the size of the determinants, respectively. The transformation $\gamma\beta$ acts on the J-matrix by $J_{m,n} \mapsto J_{m-1,n}$. Also the transformation $\gamma_2\beta\gamma_1$ with

$$\begin{aligned}\gamma_1: \qquad & J \mapsto \begin{pmatrix}1 & \\ & 1\end{pmatrix} J \begin{pmatrix} & -1\\ 1 & \end{pmatrix},\\ \gamma_2: \qquad & J \mapsto \begin{pmatrix} & 1\\ 1 & \end{pmatrix} J \begin{pmatrix}1 & \\ & -1\end{pmatrix}\end{aligned} \tag{2.16}$$

acts by $J_{m,n} \mapsto J_{m,n+1}$.

3. Reduction to the Painlevé II equation

Mason and Woodhouse have shown that the $SL(2, \mathbb{C})$ ASDYM equation can be reduced to the Painlevé equations [1, 2]. Murata has reconstructed this reduction process by using notions in the theory of generalized confluent hypergeometric functions [3]. Let us consider the ASDYM equation defined on the Grassmann variety Gr(2, 4). A Jordan group associated with a Young diagram of weight 4 naturally acts on Gr(2, 4). Suppose that the gauge potential A is invariant under the action of the Jordan group. Then the ASDYM equation is reduced to a system of ordinary differential equations, from which one can derive the Painlevé equation.

Let us summarize the derivation of the Painlevé II equation from the ASDYM equation (2.1) according to [3]. In the case of $\mathrm{P_{II}}$, we choose the Jordan group of the form

$$\begin{pmatrix}1 & a & b & c\\ & 1 & a & b\\ & & 1 & a\\ & & & 1\end{pmatrix}. \tag{3.1}$$

The above criterion leads us to the coordinate transformation

$$\tilde{z} = q - \frac{r^2}{2}, \qquad \tilde{w} = -r, \qquad z = q + \frac{r^2}{2} - t, \qquad w = p + \frac{r^3}{3} - rt, \tag{3.2}$$

or

$$p = w - (\tilde{z} - z)\tilde{w} - \frac{2}{3}\tilde{w}^3, \qquad q = \tilde{z} + \frac{1}{2}\tilde{w}^2, \qquad r = -\tilde{w}, \qquad t = \tilde{z} - z + \tilde{w}^2. \tag{3.3}$$

We rewrite the gauge potential A in the form $A = P\mathrm{d}p + Q\mathrm{d}q + R\mathrm{d}r + T\mathrm{d}t$, where P, Q, R and T depend only on t. Since it is possible to fix $T = 0$ by a gauge transformation, we have

$$A_z = -rP, \qquad A_w = P, \qquad A_{\tilde{z}} = rP + Q, \qquad A_{\tilde{w}} = -(r^2+t)P - rQ - R. \tag{3.4}$$

Substituting the result into the ASDYM equation (2.1), we get a system of ordinary differential equations

$$P' = 0, \qquad Q' = [P, R], \qquad R' = [tP + R, Q], \qquad {}' = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{3.5}$$

called *the matrix Painlevé system* [9]. Here, P, Q and R are determined up to conjugation by a constant matrix. This residual gauge freedom can be exploited to reduce P to the form

$$P = \begin{pmatrix}k & 0\\ 0 & -k\end{pmatrix}, \qquad k \neq 0, \tag{3.6}$$

when the eigenvalue of P is a non-zero constant (the semi-simple case). We then obtain for six unknowns a system of equations

$$\begin{aligned}&Q_{11}' = 0, \qquad Q_{12}' = 2kR_{12}, \qquad Q_{21}' = -2kR_{21},\\ &R_{11}' = Q_{21}R_{12} - Q_{12}R_{21},\\ &R_{12}' = 2(Q_{12}R_{11} - Q_{11}R_{12}) + 2ktQ_{12},\\ &R_{21}' = 2(Q_{11}R_{21} - Q_{21}R_{11}) - 2ktQ_{21}.\end{aligned} \tag{3.7}$$

We find that

$$\begin{aligned} l &= \operatorname{tr}(PQ) = 2kQ_{11}, \\ m &= \operatorname{tr}\left(PR - \tfrac{1}{2}Q^2\right) = 2kR_{11} - \left(Q_{11}^2 + Q_{12}Q_{21}\right), \\ n &= \operatorname{tr}(QR) = 2Q_{11}R_{11} + Q_{12}R_{21} + Q_{21}R_{12}, \end{aligned} \tag{3.8}$$

are the integral constants or conserved quantities, and that the system (3.7) essentially has three unknown variables.

By using (3.7) and (3.8) we obtain for $y = -\frac{R_{12}}{Q_{12}} = -\frac{1}{2k}(\log Q_{12})'$ and $x = R_{11}$ a system of equations

$$\begin{aligned} y' &= -2\left(x - \frac{4k^2m + l^2}{8k^3}\right) + 2k\left(y - \frac{l}{4k^2}\right)^2 - \left(2kt + \frac{m}{k} + \frac{3l^2}{8k^3}\right), \\ x' &= -4k\left(y - \frac{l}{4k^2}\right)\left(x - \frac{4k^2m + l^2}{8k^3}\right) - \frac{8k^4n - l^3 - 4k^2lm}{8k^4}. \end{aligned} \tag{3.9}$$

Applying an appropriate affine transformation to y, x and t, we see that these equations become

$$y' = -4x - y^2 + 2t, \qquad x' = 2yx - \alpha, \tag{3.10}$$

with

$$\alpha = -\frac{8k^4n - l^3 - 4k^2lm}{16k^5}, \tag{3.11}$$

which are nothing but the canonical equations for $\mathrm{P_{II}}$

$$y'' = 2y^3 - 4ty + 4\left(\alpha + \tfrac{1}{2}\right). \tag{3.12}$$

Similarly, it is possible to get, for $y_- = -\frac{R_{21}}{Q_{21}} = \frac{1}{2k}(\log Q_{21})'$ and x, another system of equations

$$\begin{aligned} y_-' &= 2\left(x - \frac{4k^2m + l^2}{8k^3}\right) - 2k\left(y_- - \frac{l}{4k^2}\right)^2 + \left(2kt + \frac{m}{k} + \frac{3l^2}{8k^3}\right), \\ x' &= 4k\left(y_- - \frac{l}{4k^2}\right)\left(x - \frac{4k^2m + l^2}{8k^3}\right) + \frac{8k^4n - 4k^2lm - l^3}{8k^4}, \end{aligned} \tag{3.13}$$

from which, by the same affine transformation as above, we also get $\mathrm{P_{II}}$

$$y_-'' = 2y_-^3 - 4ty_- + 4\left(\alpha - \tfrac{1}{2}\right). \tag{3.14}$$

Comparing (3.14) with (3.12), we see that the parameter α is replaced with $\alpha - 1$. This means that the system (3.7) reduced from the ASDYM equation (2.1) is equivalent to $\mathrm{P_{II}}$ and its Bäcklund (or Schlesinger) transformation.

Remark 3.1. The coordinate transformation $(z, w, \tilde{z}, \tilde{w}) \mapsto (p, q, r, t)$ is not uniquely determined from the above criterion. In fact, it just gives us

$$\begin{aligned} z &= q + \tfrac{1}{2}(r+f)^2 + g - t, \qquad & w &= p - (r+f)t + \tfrac{1}{3}(r+f)^3 + h, \\ \tilde{z} &= q - \tfrac{1}{2}(r+f)^2 + g, & \tilde{w} &= -(r+f), \end{aligned} \tag{3.15}$$

where f, g and h are arbitrary functions of t. Let us fix this freedom so that we have

$$q = q(\tilde{z}, \tilde{w}), \qquad r = r(\tilde{z}, \tilde{w}). \tag{3.16}$$

Then the matrices A_z and A_w are proportional to P under the gauge fixing $T = 0$, and the first equation of (2.1) is reduced to a linear equation with respect to P. In this case, we immediately get $P' = 0$. The equations for Q and R are given by

$$Q' = [P, R] + h'[P, Q], \qquad R' = [tP + R, Q] + h'[P, R]. \tag{3.17}$$

Since the terms of h' give no contribution to the system of equations for the variables x, y and y_-, one can fix $f = g = h = 0$ without loss of generality.

Before discussing the construction of special solutions, we mention the matrix H. Since the matrices A_z and A_w are proportional to the constant matrix P, it is easy to solve the linear equations (2.2) for H. For any solution to $\mathrm{P_{II}}$, we have

$$H = \begin{pmatrix} \mathrm{e}^{-k(w+\tilde{w}z)} & \\ & \mathrm{e}^{k(w+\tilde{w}z)} \end{pmatrix}, \tag{3.18}$$

up to multiplication by the matrix $\widetilde{M}$.

4. Riccati solution to $\mathrm{P_{II}}$

Murata and Woodhouse have constructed the gauge potential for the Riccati solution to each of the Painlevé equations [8, 9]. In this section, we construct the J-matrix corresponding to the Riccati solution of the Painlevé II equation.

In the case of $n = (4k^2lm + l^3)/8k^4 (\alpha = 0)$, the second equation of (3.9) admits the specialization $x = (4k^2m + l^2)/8k^3$. The first equation of (3.9) becomes the Riccati equation

$$y' = 2k\left(y - \frac{l}{4k^2}\right)^2 - \left(2kt + \frac{m}{k} + \frac{3l^2}{8k^3}\right). \tag{4.1}$$

Setting

$$y = -\frac{1}{2k}\left(\frac{\psi'}{\psi} - \frac{l}{2k}\right), \tag{4.2}$$

we get for ψ a linear equation

$$\psi'' = \left(4k^2t + 2m + \frac{3l^2}{4k^2}\right)\psi, \tag{4.3}$$

which is equivalent to Airy's differential equation $\psi'' = 2t\psi$ under the same affine transformation as above. Noting that $Q'_{12} = 2kR_{12}$, we can get

$$Q_{12} = \mathrm{e}^{-\frac{l}{2k}t}\psi, \qquad R_{12} = \frac{\mathrm{e}^{-\frac{l}{2k}t}}{2k}\left(\psi' - \frac{l}{2k}\psi\right). \tag{4.4}$$

We also obtain $Q_{21} = R_{21} = 0$ from (3.8). Then $A_{\tilde{z}}$ and $A_{\tilde{w}}$ for the Riccati solution are expressed in terms of upper triangular matrices as

$$A_{\tilde{z}} = \begin{pmatrix} -k\tilde{w} + \frac{l}{2k} & \mathrm{e}^{-\frac{l}{2k}(\tilde{z}-z+\tilde{w}^2)}\psi \\ & k\tilde{w} - \frac{l}{2k} \end{pmatrix}, \tag{4.5}$$

and

$$A_{\tilde{w}} = \begin{pmatrix} -k(\tilde{z} - z + 2\tilde{w}^2) + \frac{l}{2k}\tilde{w} - \frac{4k^2m+l^2}{8k^3} & \frac{\mathrm{e}^{-\frac{l}{2k}(\tilde{z}-z+\tilde{w}^2)}}{2k}\left(2k\tilde{w}\psi - \psi' + \frac{l}{2k}\psi\right) \\ & k(\tilde{z} - z + 2\tilde{w}^2) - \frac{l}{2k}\tilde{w} + \frac{4k^2m+l^2}{8k^3} \end{pmatrix}, \tag{4.6}$$

respectively.

It is easy to see that $\widetilde{H}$ also becomes upper triangular, by solving the linear equations (2.2). In fact, one can get

$$\widetilde{H} = \begin{pmatrix} \widetilde{H}_{11} & \widetilde{H}_{12} \\ & \widetilde{H}_{11}^{-1} \end{pmatrix} \tag{4.7}$$

with

$$\begin{aligned} \widetilde{H}_{11} &= \exp\left[\left(k\tilde{w} - \frac{l}{2k}\right)\tilde{z} + k\left(-z\tilde{w} + \frac{2}{3}\tilde{w}^3\right) - \frac{l}{4k}\tilde{w}^2 + \frac{4k^2m + l^2}{8k^3}\tilde{w}\right], \\ \widetilde{H}_{12} &= -\exp\left[\left(k\tilde{w} - \frac{l}{2k}\right)\tilde{z}\right] \\ &\quad \times \int^{\tilde{z}} \exp\left[k\tilde{w}(z - 2\tilde{z}) + \frac{l}{2k}(\tilde{z} + z) - \frac{2}{3}k\tilde{w}^3 - \frac{l}{4k}\tilde{w}^2 - \frac{4k^2m + l^2}{8k^3}\tilde{w}\right]\psi\,\mathrm{d}\tilde{z}. \end{aligned} \tag{4.8}$$

It is possible to get the J-matrix from $J = \widetilde{H}^{-1}H$. Let us choose the matrices M and $\widetilde{M}$ as

$$\begin{aligned} M &= \begin{pmatrix} \mathrm{e}^{\chi} & \\ & \mathrm{e}^{-\chi} \end{pmatrix}, & \widetilde{M} &= \begin{pmatrix} \mathrm{e}^{\tilde{\chi}} & \\ & \mathrm{e}^{-\tilde{\chi}} \end{pmatrix}, \\ \chi &= -kw, & \tilde{\chi} &= \left(k\tilde{w} - \frac{l}{2k}\right)\tilde{z} + \frac{2}{3}k\tilde{w}^3 - \frac{l}{4k}\tilde{w}^2 + \frac{4k^2m + l^2}{8k^3}\tilde{w}, \end{aligned} \tag{4.9}$$

to normalize the diagonal elements of the J-matrix to 1. Thus, we obtain

$$M^{-1}J\widetilde{M} = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix} \tag{4.10}$$

with

$$\varphi = \int^{\tilde{z}} \exp\left[2k\left(w - (\tilde{z} - z)\tilde{w} - \frac{2}{3}\tilde{w}^3\right) + \frac{l}{2k}(\tilde{z} + z) - \frac{4k^2m + l^2}{4k^3}\tilde{w}\right]\psi\,\mathrm{d}\tilde{z}. \tag{4.11}$$

What is remarkable is the following fact.

Proposition 4.1. *The function φ defined by (4.11) satisfies the Laplace equation*

$$(\partial_w\partial_{\tilde{w}} - \partial_z\partial_{\tilde{z}})\varphi = 0. \tag{4.12}$$

This means that the seed solution (2.10) to Yang's equation is reduced to the Riccati solution to $\mathrm{P_{II}}$ when we choose the particular solution (4.11) to the Laplace equation.

Proposition 4.2. *The J-matrix for the Riccati solution to P_{II} is given by*

$$J = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix}, \tag{4.13}$$

$$\begin{aligned} \varphi &= \int^{\tilde{z}} \mathrm{e}^{\eta}\psi\,\mathrm{d}\tilde{z}, \\ \eta &= 2k\left(w - (\tilde{z} - z)\tilde{w} - \frac{2}{3}\tilde{w}^3\right) + \frac{l}{2k}(\tilde{z} + z) - \frac{4k^2m + l^2}{4k^3}\tilde{w}. \end{aligned} \tag{4.14}$$

When we fix

$$\begin{aligned} H &= \begin{pmatrix} \mathrm{e}^{\xi} & \\ & \mathrm{e}^{-\xi} \end{pmatrix}, \\ \xi &= -k(w + \tilde{w}z) + \left(k\tilde{w} - \frac{l}{2k}\right)\tilde{z} + \frac{2}{3}k\tilde{w}^3 - \frac{l}{4k}\tilde{w}^2 + \frac{4k^2m + l^2}{8k^3}\tilde{w}, \end{aligned} \tag{4.15}$$

the components of the gauge potential are recovered by

$$\begin{aligned} A_z &= -\partial_z H H^{-1}, & A_w &= -\partial_w H H^{-1}, \\ A_{\tilde{z}} &= (-\partial_{\tilde{z}} H + HJ^{-1}\partial_{\tilde{z}} J)H^{-1}, & A_{\tilde{w}} &= (-\partial_{\tilde{w}} H + HJ^{-1}\partial_{\tilde{w}} J)H^{-1}. \end{aligned} \tag{4.16}$$

5. The J-matrix for the classical transcendental solutions to P_{II}

In this section, we construct the J-matrix for the classical transcendental solutions to the Painlevé II equation, based on the results of the previous section.

We have the following proposition on the classical transcendental solutions to P_{II} [11, 7].

Proposition 5.1. *Define the functions $\tau_N (N \in \mathbb{Z}_{\geqslant 0})$ by*

$$\tau_N = \begin{vmatrix} \psi^{(0)} & \psi^{(1)} & \cdots & \psi^{(N-1)} \\ \psi^{(1)} & \psi^{(2)} & \cdots & \psi^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi^{(N-1)} & \psi^{(N)} & \cdots & \psi^{(2N-2)} \end{vmatrix}, \qquad \psi^{(i)} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \psi, \tag{5.1}$$

where ψ is the general solution to Airy's differential equation (4.3). Then the bilinear relations

$$\begin{aligned} &\left[D_t^2 - \left(4k^2 t + 2m + \frac{3l^2}{4k^2}\right)\right] \tau_{N+1} \cdot \tau_N = 0, \\ &\tau_{N+1}\tau_{N-1} = \tau_N''\tau_N - (\tau_N')^2, \\ &D_t \tau_{N+1} \cdot \tau_{N-1} = 4k^2 N \tau_N^2 \end{aligned} \tag{5.2}$$

hold, and

$$y = -\frac{1}{2k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\log \frac{\tau_{N+1}}{\tau_N}\right) - \frac{l}{2k}\right], \qquad x = \frac{4k^2 m + l^2}{8k^3} - \frac{1}{2k}\frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} \tag{5.3}$$

with

$$n = \frac{4k^2 lm + l^3}{8k^4} - 2kN \tag{5.4}$$

give rise to a family of classical transcendental solutions to P_{II}.

By using (3.8) and the bilinear relations (5.2), the matrices Q and R can be calculated as

$$Q = \begin{pmatrix} \frac{l}{2k} & \frac{\mathrm{e}^{-\frac{l}{2k}t}}{(2k)^{2N}} \frac{\tau_{N+1}}{\tau_N} \\ -(2k)^{2N}\,\mathrm{e}^{\frac{l}{2k}t} \frac{\tau_{N-1}}{\tau_N} & -\frac{l}{2k} \end{pmatrix} \tag{5.5}$$

and

$$R = \begin{pmatrix} \frac{4k^2m+l^2}{8k^3} - \frac{1}{2k}\frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} & \frac{\mathrm{e}^{-\frac{l}{2k}t}}{(2k)^{2N+1}} \frac{\left(D_t - \frac{l}{2k}\right)\tau_{N+1}\cdot\tau_N}{\tau_N^2} \\ (2k)^{2N-1}\,\mathrm{e}^{\frac{l}{2k}t} \frac{\left(D_t + \frac{l}{2k}\right)\tau_{N-1}\cdot\tau_N}{\tau_N^2} & -\frac{4k^2m+l^2}{8k^3} + \frac{1}{2k}\frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} \end{pmatrix}, \tag{5.6}$$

respectively.

Because the function φ defined by (4.11) satisfies the Laplace equation, we expect that the classical transcendental solutions to P_{II} in proposition 5.1 can be obtained as a specialization of the family of solutions to Yang's equation in proposition 2.2. In fact, we have the following.

Theorem 5.2. *Define a sequence of functions $\varphi_j (j = -1, 0, 1, 2, \ldots)$ by*

$$\begin{aligned} &\varphi_{-1} = \int^{\tilde{z}} \mathrm{e}^{\eta}\psi \,\mathrm{d}\tilde{z}, \\ &\eta = 2k\left(w - (\tilde{z} - z)\tilde{w} - \frac{2}{3}\tilde{w}^3\right) + \frac{l}{2k}(\tilde{z} + z) - \frac{4k^2m + l^2}{4k^3}\tilde{w} \end{aligned} \tag{5.7}$$

with

$$\varphi_{j+1} = \frac{1}{2k}\partial_{\tilde{z}}\varphi_j, \tag{5.8}$$

and the functions τ_n^m *by* (2.11). *Then the* J*-matrix corresponding to the classical transcendental solutions of the Painlevé II equation is given by*

$$J = J_N = \frac{1}{\tau_N^{N-1}}\begin{pmatrix} \tau_N^{N-2} & \tau_{N+1}^{N-1} \\ \tau_{N-1}^{N-1} & \tau_N^N \end{pmatrix}. \tag{5.9}$$

The components of the gauge potential are recovered by (4.15) *and* (4.16).

The proof of this theorem is given in appendix A.

6. The Painlevé IV equation

By a similar discussion to the case of the Painlevé II equation, one can also construct the J-matrix for the classical transcendental solutions to the Painlevé IV equation. In this section, we present the corresponding result on P_{IV}.

6.1. Reduction to the Painlevé IV equation

The matrix Painlevé system for the Painlevé IV equation is also derived from the ASDYM equation (2.1), using the same criterion as in the case of the Painlevé II equation. Here, we use the Jordan group of the form

$$\begin{pmatrix} 1 & b & a & \\ & 1 & b & \\ & & 1 & \\ & & & c \end{pmatrix} \tag{6.1}$$

to obtain the coordinate transformations

$$\tilde{w} = \mathrm{e}^r, \qquad \tilde{z} = q\,\mathrm{e}^r, \qquad z = -q + t, \qquad w = p - \tfrac{1}{2}(-q+t)^2, \tag{6.2}$$

or

$$p = w + \frac{z^2}{2}, \qquad q = \frac{\tilde{z}}{\tilde{w}}, \qquad r = \log\tilde{w}, \qquad t = z + \frac{\tilde{z}}{\tilde{w}}. \tag{6.3}$$

The components of the gauge potential are written as

$$A_w = P, \qquad A_z = (-q+t)P, \qquad A_{\tilde{w}} = \mathrm{e}^{-r}(-qQ + R), \qquad A_{\tilde{z}} = \mathrm{e}^{-r}Q. \tag{6.4}$$

Then we get the matrix Painlevé system [9]

$$P' = 0, \qquad Q' = [P, R - tQ], \qquad R' = [R, Q], \qquad {}' = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{6.5}$$

In the semi-simple case, we obtain a system of equations

$$\begin{aligned} &Q'_{11} = 0, \\ &Q'_{12} = 2k(R_{12} - tQ_{12}), && Q'_{21} = -2k(R_{21} - tQ_{21}), \\ &R'_{11} = Q_{21}R_{12} - Q_{12}R_{21}, \\ &R'_{12} = 2(Q_{12}R_{11} - Q_{11}R_{12}), && R'_{21} = 2(Q_{11}R_{21} - Q_{21}R_{11}), \end{aligned} \tag{6.6}$$

which also has three conserved quantities

$$
\begin{aligned}
l &= \operatorname{tr}(PQ) = 2kQ_{11},\\
m^2 &= \tfrac{1}{2}\operatorname{tr}(R^2) = R_{11}^2 + R_{12}R_{21},\\
n &= \operatorname{tr}\left(PR - \tfrac{1}{2}Q^2\right) = 2kR_{11} - \left(Q_{11}^2 + Q_{12}Q_{21}\right).
\end{aligned}
\tag{6.7}
$$

Let us introduce the variables y and x by

$$
y = \frac{R_{12}}{Q_{12}} = \frac{1}{2k}(\log Q_{12})' + t, \qquad R_{11} = xy + m. \tag{6.8}
$$

Then we obtain from (6.6) and (6.7) a system of equations

$$
\begin{aligned}
y' &= 2yx - 2ky^2 + \left(2kt - \frac{l}{k}\right)y + 2m,\\
x' &= -x^2 + \left[4ky - \left(2kt - \frac{l}{k}\right)\right]x - \left(n - 2km + \frac{l^2}{4k^2}\right).
\end{aligned}
\tag{6.9}
$$

Applying an appropriate affine transformation to y, x and t, we get

$$
y' = 2yx - (y^2 + ty + \kappa_0), \qquad x' = -x^2 + (2y + t)x - \theta_\infty \tag{6.10}
$$

with

$$
\kappa_0 = 2m, \qquad \theta_\infty = -\frac{1}{2k}\left(n - 2km + \frac{l^2}{4k^2}\right), \tag{6.11}
$$

which is nothing but the canonical equations for P_{IV}

$$
\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 2ty^2 + \frac{1}{2}t^2y - (-\kappa_0 + 2\theta_\infty + 1)y - \frac{\kappa_0^2}{2y}. \tag{6.12}
$$

Similarly, it is possible to get, for $y_- = \dfrac{R_{21}}{Q_{21}} = -\dfrac{1}{2k}(\log Q_{21})' + t$ and x, another system of equations

$$
\begin{aligned}
y_-' &= -2y_-x + 2ky_-^2 - \left(2kt - \frac{l}{k}\right)y_- - 2m,\\
x' &= x^2 - \left[4ky_- \left(2kt - \frac{l}{k}\right)\right]x + \left(n - 2km + \frac{l^2}{4k^2}\right),
\end{aligned}
\tag{6.13}
$$

from which, by the same affine transformation as above, we also get P_{IV}

$$
\frac{d^2y_-}{dt^2} = \frac{1}{2y_-}\left(\frac{dy_-}{dt}\right)^2 + \frac{3}{2}y_-^3 + 2ty_-^2 + \frac{1}{2}t^2y_- - (-\kappa_0 + 2\theta_\infty - 1)y_- - \frac{\kappa_0^2}{2y_-}. \tag{6.14}
$$

Comparing (6.14) with (6.12), we see that the parameter θ_∞ is replaced with $\theta_\infty - 1$.

6.2. *Riccati solution*

Let us consider the Riccati solution to the Painlevé IV equation. It is easy to see that the system of equations (6.9) admits a specialization to $x = 0$ in the case of $n = 2km - l^2/4k^2(\theta_\infty = 0)$. Then the first equation of (6.9) becomes a Riccati equation

$$
y' = -2ky^2 + \left(2kt - \frac{l}{k}\right)y + 2m. \tag{6.15}
$$

Setting

$$y = \frac{1}{2k}\left[\frac{\psi'}{\psi} + \left(kt - \frac{l}{2k}\right)\right], \tag{6.16}$$

we get for ψ a linear equation

$$\psi'' + \left[-4km + k - \left(kt - \frac{l}{2k}\right)^2\right]\psi = 0, \tag{6.17}$$

which is equivalent to Weber's differential equation

$$\psi'' + \left(\nu + \frac{1}{2} - \frac{t^2}{4}\right)\psi = 0 \tag{6.18}$$

with $\kappa_0 = \nu + 1$, under the same affine transformation as above. Noting that $Q'_{12} = 2k(R_{12} - tQ_{12})$, we can get

$$Q_{12} = \mathrm{e}^{-\frac{1}{2k}(kt+\frac{l}{2k})^2}\psi_\nu, \qquad R_{12} = \frac{1}{2k}\,\mathrm{e}^{-\frac{1}{2k}(kt+\frac{l}{2k})^2}\left[\psi'_\nu + \left(kt - \frac{l}{2k}\right)\psi_\nu\right], \tag{6.19}$$

where $\psi = \psi_\nu$ is a general solution to (6.17). We also obtain $Q_{21} = R_{21} = 0$ from (6.7). Then $A_{\tilde{z}}$ and $A_{\tilde{w}}$ for the Riccati solution are expressed in terms of upper triangular matrices as

$$A_{\tilde{z}} = \frac{1}{\tilde{w}}\begin{pmatrix} \frac{l}{2k} & \mathrm{e}^{-\frac{1}{2k}[k(z+\frac{\tilde{z}}{\tilde{w}})+\frac{l}{2k}]^2}\psi_\nu \\ & -\frac{l}{2k} \end{pmatrix} \tag{6.20}$$

and

$$A_{\tilde{w}} = \frac{1}{\tilde{w}}\begin{pmatrix} \frac{\nu+1}{2} - \frac{l}{2k}\frac{\tilde{z}}{\tilde{w}} & \frac{1}{2k}\,\mathrm{e}^{-\frac{1}{2k}[k(z+\frac{\tilde{z}}{\tilde{w}})+\frac{l}{2k}]^2}\left[\psi'_\nu + \left(kt - \frac{l}{2k}\right)\psi_\nu - 2k\frac{\tilde{z}}{\tilde{w}}\psi_\nu\right] \\ & -\frac{\nu+1}{2} + \frac{l}{2k}\frac{\tilde{z}}{\tilde{w}} \end{pmatrix}, \tag{6.21}$$

respectively.

Solving the linear equations (2.2), we see that the matrix $\widetilde{H}$ can be expressed in the form (4.7) with

$$\begin{aligned} \widetilde{H}_{11} &= \tilde{w}^{-\frac{\nu+1}{2}}\exp\left(-\frac{l}{2k}\frac{\tilde{z}}{\tilde{w}}\right), \\ \widetilde{H}_{12} &= -\exp\left(-\frac{l}{2k}\frac{\tilde{z}}{\tilde{w}}\right)\int^{\tilde{z}}\tilde{w}^{\frac{\nu+1}{2}-1}\exp\left\{-\frac{1}{2k}\left[k\left(z + \frac{\tilde{z}}{\tilde{w}}\right) + \frac{l}{2k}\right]^2 + \frac{l}{k}\frac{\tilde{z}}{\tilde{w}}\right\}\psi_\nu\,\mathrm{d}\tilde{z}. \end{aligned} \tag{6.22}$$

The matrix H is given by

$$H = \begin{pmatrix} \mathrm{e}^{-k(w+\frac{z^2}{2})} & \\ & \mathrm{e}^{k(w+\frac{z^2}{2})} \end{pmatrix}, \tag{6.23}$$

up to multiplication by the matrix $\widetilde{M}$ for any solution to $\mathrm{P_{IV}}$. Thus, choosing the matrices M and $\widetilde{M}$ as

$$M = H, \qquad \widetilde{M} = \begin{pmatrix} \widetilde{H}_{11} & \\ & \widetilde{H}_{11}^{-1} \end{pmatrix}, \tag{6.24}$$

we obtain

$$M^{-1}J\widetilde{M} = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix} \tag{6.25}$$

with

$$\varphi = \int^{\tilde{z}} \tilde{w}^{\nu} \exp\left\{2k\left(w + \frac{z^2}{2}\right) - \frac{1}{2k}\left[k\left(z + \frac{\tilde{z}}{\tilde{w}}\right) + \frac{l}{2k}\right]^2 + \frac{l}{k}\frac{\tilde{z}}{\tilde{w}}\right\}\psi_\nu \, \mathrm{d}\tilde{z}. \tag{6.26}$$

We also find that the function φ satisfies the Laplace equation $(\partial_w\partial_{\tilde{w}} - \partial_z\partial_{\tilde{z}})\varphi = 0$.

Proposition 6.1. *The J-matrix for the Riccati solution to P_{IV} is given by*

$$J = \begin{pmatrix} 1 & \varphi \\ & 1 \end{pmatrix} \tag{6.27}$$

with

$$\varphi = \int^{\tilde{z}} \tilde{w}^{\nu}\, \mathrm{e}^{\eta}\psi_\nu \, \mathrm{d}\tilde{z}, \qquad \eta = 2k\left(w + \frac{z^2}{2}\right) - \frac{1}{2k}\left[k\left(z + \frac{\tilde{z}}{\tilde{w}}\right) + \frac{l}{2k}\right]^2 + \frac{l}{k}\frac{\tilde{z}}{\tilde{w}}. \tag{6.28}$$

When we fix

$$H = \begin{pmatrix} \tilde{w}^{-\frac{\nu+1}{2}}\,\mathrm{e}^{\xi} & \\ & \tilde{w}^{\frac{\nu+1}{2}}\,\mathrm{e}^{-\xi} \end{pmatrix}, \qquad \xi = -k\left(w + \frac{z^2}{2}\right) - \frac{l}{2k}\frac{\tilde{z}}{\tilde{w}}, \tag{6.29}$$

the components of the gauge potential are recovered by

$$\begin{aligned} A_z &= -\partial_z H H^{-1}, & A_w &= -\partial_w H H^{-1}, \\ A_{\tilde{z}} &= (-\partial_{\tilde{z}} H + H J^{-1}\partial_{\tilde{z}} J)H^{-1}, & A_{\tilde{w}} &= (-\partial_{\tilde{w}} H + H J^{-1}\partial_{\tilde{w}} J)H^{-1}. \end{aligned} \tag{6.30}$$

We remark that the general solution to (6.17) can be expressed by

$$\psi_\nu = c_1\frac{D_\nu(t)}{\Gamma(\nu+1)} + c_2\Gamma(-\nu)D_\nu(-t), \tag{6.31}$$

where $D_\nu(t)$ is the hyperbolic cylinder function, $\Gamma(\nu)$ is the Gamma function and $c_i (i = 1, 2)$ are arbitrary complex constants. Introducing the function $\phi_\nu = \mathrm{e}^{-\frac{1}{2k}(kt - \frac{l}{2k})^2}\psi_\nu$, we have

$$\varphi = \int^{\tilde{z}} \tilde{w}^{\nu}\mathrm{e}^{\widehat{\eta}}\phi_\nu \, \mathrm{d}\tilde{z}, \qquad \widehat{\eta} = 2k\left(w + \frac{z^2}{2}\right) - \frac{l}{k}z. \tag{6.32}$$

The contiguity relation of the hyperbolic cylinder function

$$D_\nu' = \left(kt - \frac{l}{2k}\right)D_\nu - (-2k)^{\frac{1}{2}}\nu D_{\nu-1} \tag{6.33}$$

leads us to $\phi_\nu' = -(-2k)^{\frac{1}{2}}\phi_{\nu-1}$. Then it is possible to explicitly calculate the integral in φ and get

$$\varphi = \int^{\tilde{z}} \tilde{w}^{\nu}\, \mathrm{e}^{\widehat{\eta}}\phi_\nu \, \mathrm{d}\tilde{z} = -(-2k)^{-\frac{1}{2}}\tilde{w}^{\nu+1}\,\mathrm{e}^{\widehat{\eta}}\phi_{\nu+1}. \tag{6.34}$$

6.3. The J-matrix for the classical transcendental solutions

We have the following proposition on the classical transcendental solutions to the Painlevé IV equation [7].

Proposition 6.2. *Define the functions $\tau_N^{(\nu)} (N \in \mathbb{Z}_{\geqslant 0})$ by*

$$\tau_N^{(\nu)} = \begin{vmatrix} \phi_\nu^{(0)} & \phi_\nu^{(1)} & \cdots & \phi_\nu^{(N-1)} \\ \phi_\nu^{(1)} & \phi_\nu^{(2)} & \cdots & \phi_\nu^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_\nu^{(N-1)} & \phi_\nu^{(N)} & \cdots & \phi_\nu^{(2N-2)} \end{vmatrix}, \qquad \phi_\nu^{(i)} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \phi_\nu. \tag{6.35}$$

Then

$$y=\frac{1}{2k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\tau_{N+1}^{(\nu)}}{\tau_N^{(\nu)}}+\left(2kt-\frac{l}{k}\right)\right]=-(-2k)^{-\frac{1}{2}}(\nu+1)\frac{\tau_N^{(\nu-1)}\tau_{N+1}^{(\nu+1)}}{\tau_N^{(\nu)}\tau_{N+1}^{(\nu)}},$$
$$x=-(-2k)^{-\frac{1}{2}}\frac{\tau_{N+1}^{(\nu)}\tau_{N-1}^{(\nu-1)}}{\tau_N^{(\nu-1)}\tau_N^{(\nu)}}, \tag{6.36}$$

with

$$m=\frac{\nu+1}{2},\qquad n=k(\nu+1)-\frac{l^2}{4k^2}-2kN, \tag{6.37}$$

give rise to a family of classical transcendental solutions to P_{IV}.

We have for $\tau_N^{(\nu)}$ the bilinear relations

$$\begin{aligned}
\tau_{N+1}^{(\nu)}\tau_{N-1}^{(\nu-1)}&=-(-2k)^{-\frac{1}{2}}D_t\tau_N^{(\nu-1)}\cdot\tau_N^{(\nu)},\\
\tau_N^{(\nu)}\tau_{N+1}^{(\nu-1)}&=-(-2k)^{-\frac{1}{2}}D_t\tau_{N+1}^{(\nu)}\cdot\tau_N^{(\nu-1)},\\
(\nu+1)\tau_N^{(\nu-1)}\tau_{N+1}^{(\nu+1)}&=(-2k)^{-\frac{1}{2}}\left[D_t+\left(2kt-\tfrac{l}{k}\right)\right]\tau_{N+1}^{(\nu)}\cdot\tau_N^{(\nu)}
\end{aligned} \tag{6.38}$$

and

$$\begin{aligned}
(-2k)^{-1}\tau_{N+1}^{(\nu+1)}\tau_{N-1}^{(\nu-1)}&=\tau_N^{(\nu+1)}\tau_N^{(\nu-1)}-\tau_N^{(\nu)}\tau_N^{(\nu)},\\
(\nu+1)\tau_{N+1}^{(\nu+1)}\tau_{N-1}^{(\nu-1)}&=\tau_{N+1}^{(\nu)}\tau_{N-1}^{(\nu)}+2kN\tau_N^{(\nu)}\tau_N^{(\nu)}.
\end{aligned} \tag{6.39}$$

Then the matrices Q and R can be expressed as

$$Q=\begin{pmatrix}\frac{l}{2k} & \frac{1}{(2k)^{2N}}\,\mathrm{e}^{-\frac{l}{k}t}\frac{\tau_{N+1}^{(\nu)}}{\tau_N^{(\nu)}}\\ -(2k)^{2N}\,\mathrm{e}^{\frac{l}{k}t}\frac{\tau_{N-1}^{(\nu)}}{\tau_N^{(\nu)}} & -\frac{l}{2k}\end{pmatrix} \tag{6.40}$$

and

$$R=\begin{pmatrix}\frac{\nu+1}{2}-\frac{1}{2k}(\nu+1)\frac{\tau_{N+1}^{(\nu+1)}\tau_{N-1}^{(\nu-1)}}{\tau_N^{(\nu)}\tau_N^{(\nu)}} & \frac{1}{(2k)^{2N+1}}\,\mathrm{e}^{-\frac{l}{k}t}\frac{\left[D_t+\left(2kt-\frac{l}{k}\right)\right]\tau_{N+1}^{(\nu)}\cdot\tau_N^{(\nu)}}{\tau_N^{(\nu)}\tau_N^{(\nu)}}\\ (2k)^{2N-1}\,\mathrm{e}^{\frac{l}{k}t}\frac{\left[D_t-\left(2kt-\frac{l}{k}\right)\right]\tau_{N-1}^{(\nu)}\cdot\tau_N^{(\nu)}}{\tau_N^{(\nu)}\tau_N^{(\nu)}} & -\frac{\nu+1}{2}+\frac{1}{2k}(\nu+1)\frac{\tau_{N+1}^{(\nu+1)}\tau_{N-1}^{(\nu-1)}}{\tau_N^{(\nu)}\tau_N^{(\nu)}}\end{pmatrix}, \tag{6.41}$$

respectively.

Theorem 6.3. *Define a sequence of functions* φ_j *by*

$$\varphi_{-1}=\int^{\tilde z}\tilde w^{\nu}\,\mathrm{e}^{\hat\eta}\phi_\nu\,\mathrm{d}\tilde z=-(-2k)^{-\frac{1}{2}}\tilde w^{\nu+1}\,\mathrm{e}^{\hat\eta}\phi_{\nu+1} \tag{6.42}$$

and $\varphi_{j+1}=\frac{1}{2k}\partial_{\tilde z}\varphi_j$, *and define the functions* τ_n^m *by (2.11). Then the* J*-matrix for the classical transcendental solutions to the Painlevé IV equation is given by*

$$J=J_N^{(\nu)}=\frac{1}{\tau_N^{N-1}}\begin{pmatrix}\tau_N^{N-2} & \tau_{N+1}^{N-1}\\ \tau_{N-1}^{N-1} & \tau_N^{N}\end{pmatrix}. \tag{6.43}$$

The components of the gauge potential are recovered by (6.30) with

$$H=\begin{pmatrix}\tilde w^{-\frac{\nu+1}{2}+N}\,\mathrm{e}^{\xi} & \\ & \tilde w^{\frac{\nu+1}{2}-N}\,\mathrm{e}^{-\xi}\end{pmatrix}. \tag{6.44}$$

The proof of this theorem is given in appendix B.

7. A remark on the Bäcklund transformations

It is well known that the Painlevé II and IV equations admit the affine Weyl group symmetries of types $A_1^{(1)}$ and $A_2^{(1)}$, respectively, as the group of Bäcklund transformations. As for the translation subgroup, it is possible to find the corresponding transformation on the J-matrix. In the case of $\mathrm{P_{II}}$, the transformation $(\gamma_1\beta)^2$ acts on J_N by $J_N \mapsto J_{N-1}$. In the case of $\mathrm{P_{IV}}$, the transformations $\gamma_2\beta\gamma_1$ and $\beta\gamma$ act on $J_N^{(\nu)}$ by $J_N^{(\nu)} \mapsto J_{N+1}^{(\nu)}$ and $J_N^{(\nu)} \mapsto J_N^{(\nu-1)}$, respectively.

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Appendix A. A proof of theorem 5.2

What we must prove is the following relations:

$$\left(\partial_{\tilde{z}}\tau_N^N\right)\tau_N^{N-2} - \left(\partial_{\tilde{z}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1} - \left(\partial_{\tilde{z}}\tau_N^{N-1}\right)\tau_N^{N-1} = 0, \tag{A.1}$$

$$\frac{D_{\tilde{z}}\tau_{N+1}^{N-1}\cdot\tau_N^N}{\left(\tau_N^{N-1}\right)^2} = (2k)^{-2N}\,\mathrm{e}^{\eta}\,\frac{\tau_{N+1}}{\tau_N}, \tag{A.2}$$

$$\frac{D_{\tilde{z}}\tau_N^{N-2}\cdot\tau_{N-1}^{N-1}}{\left(\tau_N^{N-1}\right)^2} = (2k)^{2N}\,\mathrm{e}^{-\eta}\,\frac{\tau_{N-1}}{\tau_N} \tag{A.3}$$

and

$$\frac{D_{\tilde{w}}\tau_{N+1}^{N-1}\cdot\tau_N^N}{\left(\tau_N^{N-1}\right)^2} = (2k)^{-2N-1}\,\mathrm{e}^{\eta}\left[-\frac{\left(D_t-\frac{l}{2k}\right)\tau_{N+1}\cdot\tau_N}{\tau_N^2} + 2k\tilde{w}\frac{\tau_{N+1}}{\tau_N}\right], \tag{A.4}$$

$$\frac{D_{\tilde{w}}\tau_N^{N-2}\cdot\tau_{N-1}^{N-1}}{\left(\tau_N^{N-1}\right)^2} = -(2k)^{2N-1}\,\mathrm{e}^{-\eta}\left[\frac{\left(D_t-\frac{l}{2k}\right)\tau_N\cdot\tau_{N-1}}{\tau_N^2} - 2k\tilde{w}\frac{\tau_{N-1}}{\tau_N}\right], \tag{A.5}$$

$$\frac{\left(\partial_{\tilde{w}}\tau_N^N\right)\tau_N^{N-2} - \left(\partial_{\tilde{w}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1} - \left(\partial_{\tilde{w}}\tau_N^{N-1}\right)\tau_N^{N-1}}{\left(\tau_N^{N-1}\right)^2} = -\frac{1}{2k}\frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2}. \tag{A.6}$$

The first relation (A.1) is proved as follows. Let us introduce the notation

$$j = \begin{pmatrix}\varphi_j\\ \varphi_{j+1}\\ \vdots\end{pmatrix}. \tag{A.7}$$

For example, τ_N^N is written as $\tau_N^N = |1, 2, \ldots, N|$. Also, by using (5.8) we have

$$\begin{aligned}\partial_{\tilde{z}}\tau_N^N &= 2k|1,\ldots,N-1,N+1|,\\ \partial_{\tilde{z}}\tau_{N+1}^{N-1} &= 2k|-1,\ldots,N-2,N|,\\ \partial_{\tilde{z}}\tau_N^{N-1} &= 2k|0,\ldots,N-2,N|.\end{aligned} \tag{A.8}$$

Set $D := |-1,\ldots,N-2,N|$. Then the bilinear relation (A.1) is reduced to Jacobi's identity

$$D\cdot D\begin{bmatrix}1 & N+1\\ 1 & N+1\end{bmatrix} = D\begin{bmatrix}1\\ 1\end{bmatrix}D\begin{bmatrix}N+1\\ N+1\end{bmatrix} - D\begin{bmatrix}1\\ N+1\end{bmatrix}D\begin{bmatrix}N+1\\ 1\end{bmatrix}, \tag{A.9}$$

where $D\left[\begin{smallmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{smallmatrix}\right]$ is the minor obtained by deleting the rows with indices $i_1, \ldots, i_k$ and the columns with indices $j_1, \ldots, j_k$.

Next we prove the relations (A.2) and (A.3). By using the second bilinear relation of (2.15) and $\partial_w \varphi_j = 2k\varphi_j$, we have

$$\begin{aligned} &D_{\tilde{z}}\tau_{N+1}^{N-1} \cdot \tau_N^N = D_w \tau_{N+1}^N \cdot \tau_N^{N-1} = 2k\tau_{N+1}^N \tau_N^{N-1}, \\ &D_{\tilde{z}}\tau_N^{N-2} \cdot \tau_{N-1}^{N-1} = D_w \tau_N^{N-1} \cdot \tau_{N-1}^{N-2} = 2k\tau_N^{N-1}\tau_{N-1}^{N-2}. \end{aligned} \tag{A.10}$$

It is easy to see that

$$\tau_N^{N-1} = \frac{e^{N\eta}}{(2k)^{N^2}}\tau_N. \tag{A.11}$$

Then we get (A.2) and (A.3). The relations (A.4) and (A.5) can be proved in a similar way. In fact, the first bilinear relation of (2.15) and (A.11) leads us to

$$\begin{aligned} &\frac{D_{\tilde{w}}\tau_{N+1}^{N-1} \cdot \tau_N^N}{\left(\tau_N^{N-1}\right)^2} = \frac{D_z \tau_{N+1}^N \cdot \tau_N^{N-1}}{\left(\tau_N^{N-1}\right)^2} = (2k)^{-2N-1}\partial_z\left(e^{\eta}\frac{\tau_{N+1}}{\tau_N}\right), \\ &\frac{D_{\tilde{w}}\tau_N^{N-2} \cdot \tau_{N-1}^{N-1}}{\left(\tau_N^{N-1}\right)^2} = \frac{D_z \tau_N^{N-1} \cdot \tau_{N-1}^{N-2}}{\left(\tau_N^{N-1}\right)^2} = -(2k)^{2N-1}\partial_z\left(e^{-\eta}\frac{\tau_{N-1}}{\tau_N}\right). \end{aligned} \tag{A.12}$$

Let us prove the relation (A.6). By using $\partial_{\tilde{w}}\varphi_j = -2k\varphi_{j+2} + (l/k)\varphi_{j+1}$, we have

$$\begin{aligned} \partial_{\tilde{w}}\tau_N^N &= 2k|1, \ldots, N-2, N, N+1| \\ &\quad - 2k|1, \ldots, N-1, N+2| + \frac{l}{k}|1, \ldots, N-1, N+1|, \\ \partial_{\tilde{w}}\tau_{N+1}^{N-1} &= 2k|-1, \ldots, N-3, N-1, N| \\ &\quad - 2k|-1, \ldots, N-2, N+1| + \frac{l}{k}|-1, \ldots, N-2, N|, \\ \partial_{\tilde{w}}\tau_N^{N-1} &= 2k|0, \ldots, N-3, N-1, N| \\ &\quad - 2k|0, \ldots, N-2, N+1| + \frac{l}{k}|0, \ldots, N-2, N|. \end{aligned} \tag{A.13}$$

We find that the contributions from the third terms of the right-hand sides of these relations vanish, due to Jacobi's identity (A.9) with $D := |-1, \ldots, N-2, N|$. Also the contributions from the second terms vanish, due to Jacobi's identity (A.9) with $D := |-1, \ldots, N-2, N+1|$. Then we have

$$\begin{aligned} &\left[\left(\partial_{\tilde{w}}\tau_N^N\right)\tau_N^{N-2} - \left(\partial_{\tilde{w}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1} - \left(\partial_{\tilde{w}}\tau_N^{N-1}\right)\tau_N^{N-1}\right]/(2k) \\ &\qquad = |1, \ldots, N-2, N, N+1| \times |-1, \ldots, N-2| \\ &\qquad\quad - |-1, \ldots, N-3, N-1, N| \times |1, \ldots, N-1| \\ &\qquad\quad - |0, \ldots, N-3, N-1, N| \times |0, \ldots, N-1|. \end{aligned} \tag{A.14}$$

Moreover, Jacobi's identity (A.9) with $D := |-1, \ldots, N-3, N-1, N|$ can be exploited to get

$$\begin{aligned} &\left(\text{right-hand side of (A.14)}\right) \times |1, \ldots, N-2, N| \\ &\qquad = |1, \ldots, N-2, N, N+1|(|1, \ldots, N-2, N| \times |-1, \ldots, N-2| \\ &\qquad\quad - |-1, \ldots, N-3, N-1| \times |1, \ldots, N-1|) \\ &\qquad\quad + |0, \ldots, N-3, N-1, N|(|0, \ldots, N-2, N| \times |1, \ldots, N-1| \\ &\qquad\quad - |1, \ldots, N-2, N| \times |0, \ldots, N-1|). \end{aligned} \tag{A.15}$$

One can rewrite the right-hand side of (A.15) as

$$|0,\ldots,N-2|(-|1,\ldots,N-2,N,N+1|\times|0,\ldots,N-1| \\ +|0,\ldots,N-3,N-1,N|\times|1,\ldots,N|), \tag{A.16}$$

by using appropriate Plücker relations. Due to Jacobi's identity

$$D\cdot D\begin{bmatrix}1 & N+1\\ N-1 & N+1\end{bmatrix}=D\begin{bmatrix}1\\ N-1\end{bmatrix}D\begin{bmatrix}N+1\\ N+1\end{bmatrix}-D\begin{bmatrix}1\\ N+1\end{bmatrix}D\begin{bmatrix}N+1\\ N-1\end{bmatrix} \tag{A.17}$$

with $D:=|0,\ldots,N|$, we finally obtain

$$\left(\partial_{\tilde{w}}\tau_N^N\right)\tau_N^{N-2}-\left(\partial_{\tilde{w}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1}-\left(\partial_{\tilde{w}}\tau_N^{N-1}\right)\tau_N^{N-1} \\ =2k(-|0,\ldots,N|\times|0,\ldots,N-2|)=-2k\tau_{N+1}^N\tau_{N-1}^{N-2}, \tag{A.18}$$

which yields (A.6) by using (A.11).

Appendix B. A proof of theorem 6.3

Note that we have the linear relations

$$\begin{aligned}\partial_w\varphi_j&=2k\varphi_j,\\ \partial_{\tilde{w}}\varphi_j&=2k\tilde{w}\varphi_{j+2}+(\partial_z\widehat{\eta})\varphi_{j+1}.\end{aligned} \tag{B.1}$$

In a similar way to the case of the Painlevé II equation, one can prove the following relations:

$$\left(\partial_{\tilde{z}}\tau_N^N\right)\tau_N^{N-2}-\left(\partial_{\tilde{z}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1}-\left(\partial_{\tilde{z}}\tau_N^{N-1}\right)\tau_N^{N-1}=0, \tag{B.2}$$

$$\frac{D_{\tilde{z}}\tau_{N+1}^{N-1}\cdot\tau_N^N}{\left(\tau_N^{N-1}\right)^2}=(2k)^{-2N}\tilde{w}^{\nu-2N}\,\mathrm{e}^{\widehat{\eta}}\,\frac{\tau_{N+1}^{(\nu)}}{\tau_N^{(\nu)}}, \tag{B.3}$$

$$\frac{D_{\tilde{z}}\tau_N^{N-2}\cdot\tau_{N-1}^{N-1}}{\left(\tau_N^{N-1}\right)^2}=(2k)^{2N}\tilde{w}^{-\nu+2N-2}\,\mathrm{e}^{-\widehat{\eta}}\,\frac{\tau_{N-1}^{(\nu)}}{\tau_N^{(\nu)}} \tag{B.4}$$

and

$$\frac{D_{\tilde{w}}\tau_{N+1}^{N-1}\cdot\tau_N^N}{\left(\tau_N^{N-1}\right)^2}=(2k)^{-2N-1}\tilde{w}^{\nu-2N}\,\mathrm{e}^{\widehat{\eta}}\left[\frac{\left(D_t+2kt-\frac{l}{k}\right)\tau_{N+1}^{(\nu)}\cdot\tau_N^{(\nu)}}{\left(\tau_N^{(\nu)}\right)^2}-2kq\frac{\tau_{N+1}^{(\nu)}}{\tau_N^{(\nu)}}\right], \tag{B.5}$$

$$\frac{D_{\tilde{w}}\tau_N^{N-2}\cdot\tau_{N-1}^{N-1}}{\left(\tau_N^{N-1}\right)^2}=(2k)^{2N-1}\tilde{w}^{-\nu+2N-2}\,\mathrm{e}^{-\widehat{\eta}}\left[\frac{\left(D_t+2kt-\frac{l}{k}\right)\tau_N^{(\nu)}\cdot\tau_{N-1}^{(\nu)}}{\left(\tau_N^{(\nu)}\right)^2}-2kq\frac{\tau_{N-1}^{(\nu)}}{\tau_N^{(\nu)}}\right], \tag{B.6}$$

$$\frac{\left(\partial_{\tilde{w}}\tau_N^N\right)\tau_N^{N-2}-\left(\partial_{\tilde{w}}\tau_{N+1}^{N-1}\right)\tau_{N-1}^{N-1}-\left(\partial_{\tilde{w}}\tau_N^{N-1}\right)\tau_N^{N-1}}{\left(\tau_N^{N-1}\right)^2}=\frac{1}{2k\tilde{w}}\frac{\tau_{N+1}^{(\nu)}\tau_{N-1}^{(\nu)}}{\left(\tau_N^{(\nu)}\right)^2}, \tag{B.7}$$

where we use

$$\tau_N^{N-1}=\frac{1}{(2k)^{N^2}}\tilde{w}^{N(\nu-N+1)}\,\mathrm{e}^{N\widehat{\eta}}\tau_N^{(\nu)}. \tag{B.8}$$

The components of the gauge potential are recovered from (6.30) by using the above relations and the second relation of (6.39).

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